Five Dimensional Cosmological Models in General Relativity

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Abstract

A Five dimensional Kaluza-Klein space-time is considered in the presence of a perfect fluid source with variable G and Λ. An expanding universe is found by using a relation between the metric potential and an equation of state. The gravitational constant is found to decrease with time as $G \sim t^{-(1-\omega)}$ whereas the variation for the cosmological constant follows as $\Lambda \sim t^{-2}$, $\Lambda \sim (\dot{R}/R)^2$ and $\Lambda \sim \ddot{R}/R$ where ω is the equation of state parameter and R is the scale factor.

Key words: variable G and Λ, Kaluza-Klein model, phenomenological cosmology.
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1 Introduction

In recent years there has been considerable interest in the cosmological models with variable gravitational constant G and the cosmological constant Λ. Variation of the gravitational constant was first suggested by Dirac [1] in an attempt to understand the appearance of certain very large numbers, when atomic and cosmic worlds are compared. He postulates that the gravitational constant G decreases inversely with cosmic time. Canuto et al. [2,3] made numerous suggestions based on different arguments that G is indeed time dependent. Beesham [4] has studied the creation with variable G and pointed

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out the variation of the form $G \sim t^{-1}$, originally proposed by Dirac [1]. On the other hand, Einstein introduced the cosmological constant Λ to account for a stable static universe as appeared to him at the time. When he later knew of the universal expansion he regretted its inclusion in his field equations. Now cosmologist believe that is not identically, but very close to zero. They relate this constant to the vacuum energy that first inflated a universe causing it to expand [5]. From the point of view of particle physics a vacuum energy could correspond to quantum field that is diluted to its present small value. However, other cosmologists dictate a time variation of this constant in order to account for its present smallness [6]. The variation of its constant could resolve some of the standard model problems like G, the constant Λ is a gravity coupling and both should therefore be treated on an equal footing.

The generalized Einstein's theory of gravitation with time dependent G and Λ has been proposed by Lau [7]. The possibility of variable G and Λ in Einstein theory has also been studied by Dersarkissian [8]. This relation plays an important role in cosmology. Berman [9] and Sistero [10] have considered the Einstein field equations with perfect fluid and variable G and for Robertson-Walker line element. Kalligas et al. [11] have studied FRW models with variable Λ and G and discussed the possible connection with power-law time dependence of G. Abdussattar and Vishwakarama [12] presented R-W models with variable Λ and G by admitting a contracted Ricci collineation along the fluid flow vector. Recently some of us and others have studied cosmological models with variable G and Λ in a diversified fields [13,14,15,16,17,18,19,20,21,22,23,24]. Thus the implication of time varying Λ and G are important to study the early evolution of the universe.

In the present paper a five dimensional Kaluza-Klein cosmological model is considered with variable G and Λ which provides an expanding universe. It is found that the gravitational constant decreases with time as $G \sim t^{-(1-\omega)}$ whereas the cosmological constant decreases as $\Lambda \sim t^{-2}$, $\Lambda \sim (\dot{R}/R)^2$ and $\Lambda \sim \ddot{R}/R$ where ω is the equation of state parameter, R is the scale factor. However, we have come across with some awkward situations in connection to the deceleration parameter and the age of the Universe as far as the scenario of present accelerating Universe is concerned. In the present investigation, our approach is similar to that of Collins et al. [25] for solving the Einstein field equations. The paper is organized as follows: In Sections 2 and 3 we present the basic field equations governing the models and their solutions. A critical discussion and conclusion are provided in Section 4.

2 The Einstein field equations for the cosmological model

We consider five dimensional Kaluze-Klein space-time given by

2

$$ds^2 = dt^2 - A^2(dx^2 + dy^2 + dz^2) - B^2 d\psi^2 \tag{1}$$

where A and B are function of the temporal coordinate t only.

The universe is assumed to be filled with distribution of matter represented by energy-momentum tensor for a perfect fluid

$$T_{ij} = (\rho + p)v_i v_i - p g_{ij} \tag{2}$$

where ρ is the energy density of the cosmic matter and p is its pressure, v_i is the unit flow vector such that $v_i v^i = 1$.

We assume that the matter content obeys an equation of state

$$p = \omega\rho, \ 0 \leq \omega \leq 1. \tag{3}$$

The field equations are those of Einstein but with time dependent cosmological and gravitational constants and are given by Weinberg [26]

$$R^{ij} - \frac{1}{2}Rg^{ij} = 8\pi G T^{ij} - \Lambda g^{ij}. \tag{4}$$

The spatial average scale factor $R(t)$ is given by

$$R^4 = A^3 B \tag{5}$$

and average volume scale factor $V = R^4$.

The average Hubble parameter H may be generalized in anisotropic cosmological model as

$$H = \frac{1}{4}\dot{X} = \dot{Y} = \frac{1}{4}\left(\frac{3\dot{A}}{A} + \frac{\dot{B}}{B}\right) \tag{6}$$

where $X = logV$ and $Y = logR$ and an over head dot denotes ordinary differentiation with respect to cosmic time t. We also have

$$H = \frac{1}{4}(H_1 + H_2 + H_3 + H_4) \tag{7}$$

where $H_1 = H_2 = H_3 = \dot{A}/A$ and $H_4 = \dot{B}/B$ are Hubble's factor in the directions of x, y, z and ψ respectively.

For the line element (1) the Einstein field equations (4) yield the following equations

$$\frac{3\dot{A}^2}{A^2} + \frac{3\dot{A}\dot{B}}{AB} = 8\pi G\rho + \Lambda, \tag{8}$$

$$\frac{2\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{A}^2}{A^2} = -8\pi Gp + \Lambda, \tag{9}$$

$$\frac{3\ddot{A}}{A} + \frac{3\dot{A}^2}{A^2} = -8\pi Gp + \Lambda. \tag{10}$$

If we use the equivalent energy conservation of general relativity by taking the covariant derivative of Einstein field equations (4) we find that

$$\dot{\rho} + (\rho + p)\left(\frac{3\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \frac{\dot{G}}{G}\rho + \frac{\dot{\Lambda}}{8\pi G} = 0. \tag{11}$$

This provides [17,22]

$$\dot{\rho} + (\rho + p)\left(\frac{3\dot{A}}{A} + \frac{\dot{B}}{B}\right) = 0 \tag{12}$$

as well as

$$\dot{\Lambda} = -8\pi\dot{G}\rho \tag{13}$$

which will be satisfied by the solutions of the field equations as can be verified easily.

3   The general solutions to the field equations

The equations (3), (8) - (10) and (12) are five independent equations with six unknowns A, B, ρ, p, G and Λ. Hence to get a realistic solution we assume that the expansion scalar θ in the model is proportional to the shear σ [21]. This condition leads to the relation between metric potential [25]

$$B = A^n \tag{14}$$

where $n(\neq 1)$ is a constant.

From equations (9) and (10), we have

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{k_1}{A^3 B} \tag{15}$$

where k_1 is a constant of integration. Using equation (14) in equation (15) and then integrating it we get the lime element (1) as

$$\begin{aligned}
ds^2 = dt^2 &- \left[k_1 \frac{(n+3)}{(1-n)} t + k_2(n+3) \right]^{2/(n+3)} (dx^2 + dy^2 + dz^2) \\
&- \left[k_1 \frac{(n+3)}{(1-n)} t + k_2(n+3) \right]^{2n/(n+3)} d\psi^2.
\end{aligned} \tag{16}$$

By suitable changes of constants the line element (1) can be written as

$$ds^2 = dt^2 - (at+b)^{2/(n+3)}(dx^2 + dy^2 + dz^2) - (at+b)^{2n/(n+3)} d\psi^2 \tag{17}$$

where $a = k_1(n+3)/(1-n)$ and $b = k_2(n+3)$ are two positive constants.

For the model (17), the spatial volume V, matter density ρ, pressure p, gravitational parameter G, cosmological parameter Λ are given by

$$V = R^4 = (at+b), \tag{18}$$

$$\rho = \frac{k_2}{(at+b)^{(1+\omega)}}, \tag{19}$$

$$p = \frac{\omega k_2}{(at+b)^{(1+\omega)}}, \tag{20}$$

$$G = \frac{3a^2(n+1)}{4\pi k_2(1+\omega)(n+3)^2(at+b)^{(1-\omega)}}, \tag{21}$$

$$\Lambda = \frac{3(1-\omega)a^2(n+1)}{(1+\omega)(n+3)^2(at+b)^2}. \tag{22}$$

The expansion scalar θ and shear σ are

$$\theta = 3H = \frac{a}{(at+b)}, \tag{23}$$

$$\sigma = \frac{\alpha}{(at+b)} \tag{24}$$

where α is a positive constant.

The critical density ρ_c, vacuum density ρ_Λ and density parameter Ω are given by

$$\rho_c = \frac{3H^2}{8\pi G} = \frac{(1+\omega)(n+3)^2}{32k_2(n+1)(at+b)^{(1+\omega)}}, \tag{25}$$

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} = \frac{k_2(\omega-1)}{2(at+b)^{(1+\omega)}}, \tag{26}$$

$$\Omega = \frac{8\pi G\rho}{3H^2} = \frac{(n+1)}{(1+\omega)(n+3)^2}. \tag{27}$$

Keeping in mind the form of the solution (18), i.e. $R^4 = (at+b)$, one can express all the above parameters as function of the scale factor R as well which we will follow in the following two sub-cases.

3.1 Special case studies

3.1.1 The radiation era

For the radiation epoch, we have $\omega = 1/3$. In this case

$$\rho = k_2 R^{-4/3}, \tag{28}$$

$$G = \frac{9a^2(n+1)}{16\pi K_2(n+3)^2} R^{-2/3}, \tag{29}$$

$$\Lambda = \frac{-3a^2(n+1)}{2(n+3)^2} R, \tag{30}$$

$$\rho_\nu = \frac{-k_2}{3} R^{-4/3}, \tag{31}$$

$$\rho_c = \frac{(n+3)^2}{24(n+1)} R^{-4/3}. \tag{32}$$

We observe that in this phase matter density is three times the vacuum density. The cosmic density parameter Ω in this phase is

$$\Omega = \frac{3(n+1)}{4(n+3)^2}. \tag{33}$$

3.1.2   The dust epoch

For the phase dominated by dust matters, we have $\omega = 0$. In this epoch

$$\rho = k_2 R^{-1}, \tag{34}$$

$$G = \frac{3a^2(n+1)}{4\pi k_2(n+3)^2} R^{-1}, \tag{35}$$

$$|\Lambda| = \frac{3a^2(n+1)}{(n+3)^2} R^{-2}, \tag{36}$$

$$\rho_\nu = \frac{k_2}{2} R^{-1}, \tag{37}$$

$$\rho_c = \frac{(n+3)^2}{32(n+1)} R^{-1}. \tag{38}$$

$$\frac{|\Lambda|}{8\pi G\rho} = \frac{1}{2}. \tag{39}$$

4   Discussion and conclusion

In the present phenomenological study five dimensional Kaluza-Klein cosmological models with varying G and Λ are obtained. The models obtained present an expansion scalar θ bearing a constant ratio to the anisotropy in the direction of space like unit vector λ' and with equation of state $p = \omega\rho$, $0 \le \omega \le 1$. If we inspect the solution set then it can be observed from the equation (18) that the spatial volume V is zero at $t = t_o$ where $t_o = -b/a$ and expansion scalar θ is infinite at $t = t_o$ which shows that the Universe starts evolving with zero volume and infinite rate of expansion at $t = t_o$. The scale factors also vanish at $t = t_o$ and hence the model has a *point type* singularity at the initial epoch. Since σ/θ is constant, as evident from the solutions (23) and (24), the anisotropy does not die out asymptotically. The model admits a negative Λ unless $\omega = 1$ and these type of models for the Universe with negative Λ are available in the literature [27]. We observe from the general solution set that for the stiff fluid ($\omega = 1$) the model reduces to standard Kaluza-Klein type Zel'dovich universe [28] with $\Lambda = 0$, $G = $ constant and $\rho = p \sim t^{-2}$ for the limiting case $b \to 0$ (Figs. 1 & 2). At $t = t_o$ limit all the parameters G, Λ, σ^2 and θ are infinite in value, whereas all these become zero as $t \to \infty$. The ratio between cosmic matter density parameter $\Omega_m (= 8\pi G\rho/3H^2)$ and cosmic vacuum-energy density parameter $\Omega_\Lambda (= \Lambda/3H^2)$ scales as

$$\frac{\Omega_m}{\Omega_\Lambda} = \frac{\Lambda}{8\pi G\rho} = \frac{\omega - 1}{2}, \tag{40}$$

whereas cosmic vacuum-energy density parameter can be given as

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} = \frac{q(n+1)(\omega - 1)}{(n+3)^2(1+\omega)}. \tag{41}$$

The above two expressions related to cosmic matter and vacuum density parameters do vanish for the stiff fluid condition $\omega = 1$.

If we look at the solution expressed in the equation (21) then we can observe that in general $G \sim t^{-(1-\omega)}$. Thus for the dust case ($\omega = 0$) this case is similar to the result of Dirac [1], Canuto et al. [2,3] and Beesham [4]. On the other hand, equations (22) and (23) at once imply that $\Lambda \sim H^2$ for any constant values of ω and n. Again, from the solutions (18) and (22) one can

find $\Lambda \sim \ddot{R}/R$. On the other hand for the limiting case $b \to 0$ the equation (22) yields the relation $\Lambda \sim t^{-2}$. Thus we obtain $\Lambda \sim H^2 = (\dot{R}/R)^2$, $\Lambda \sim \ddot{R}/R$ and $\Lambda \sim t^{-2}$ which is in accordance with the main dynamical laws one finds in literature proposed for the decay of Λ. Note that we find all these phenomenological relations for Λ directly from the solution set which generally consider as ad hoc basis in cosmological research. The dynamical law $\Lambda \sim (\dot{R}/R)^2$, has been proposed by Carvalho et al. [29] and considered by Salim and Waga [30], Arbab and Abdel-Rahman [31], Wetterich [32], Arbab [33] and very recently by Ray and his collaborators as well as Pradhan and his collaborators [15,16,34,18,35,36]. The decay law of the form $\Lambda \sim \ddot{R}/R$ has been considered by Arbab [37], Khadekar et al. [14], Ray et al. [15], Ray and Mukhopadhyay [34] and Ray et al. [35]. The dynamical law $\Lambda \sim t^{-2}$ has been considered by several authors, e.g. Bertolami [38], Berman and Som [39], Berman [40], Beesham [41], Pradhan et al. [13] to mention a few. We observe that $|\Lambda|$ decays faster than G where as p, ρ, ρ_ν, ρ_c scales $R^{-(\omega+1)}$ as can be seen from the Figs. 1 - 4. In the model we see that the quantity G satisfies the condition for a Machian cosmological solution i.e. $G\rho \sim H^2$ [42].

However, as far as the scenario of present accelerating Universe is concern our models seems suffer a lot in connection to the different physical parameters, such as the deceleration parameter and the age of the Universe. The deceleration parameter q for the present model is $q_0 = 3$ and hence does not correspond to the observational result $q_0 \approx -0.5$ [43,44]. However, we have observed that for the scale factor of the form $a(t) = t^n$ we can have a negative deceleration parameter when $n > 1$. In our present case $n = 1/4$ which yields $q = +ve$. Here the age of the universe is given by $t_0 = \frac{1}{3}H_0^{-1}$ which is also differs from the present estimate $t_0 = H_0^{-1} \approx 14$ Gyr [45,46,47,48]. This value, therefore, obviously suffers from the low age problem [49] as even the oldest globular clusters show a value which is $t \approx 11$ Gyr [50,51]. In this connection it is argued that dark energy models without Λ suffer from low age problem [52,49] though even in the presence of Λ we are not in a hopeful position. One of the reasons of these discrepancies may be due to the constant nature of the equation of state parameter ω which can not accommodate with the present accelerating phase of the Universe. However, this needs further investigation.

Acknowledgments

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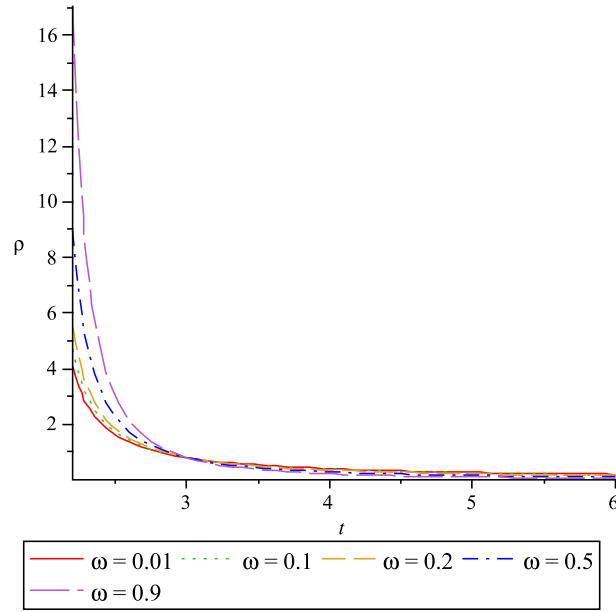[52] Sahni, V., Starobinsky, A.: Int. J. Mod. Phys. D **9**, 373 (2000)

Fig. 1. Variation of matter density with time for the given values of ω under the constraint $k_2 = 0.8$, $a = 1$ and $b = -2$.
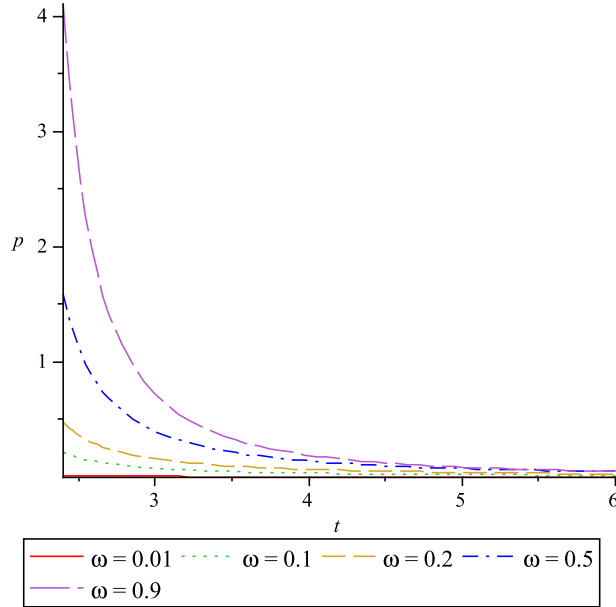

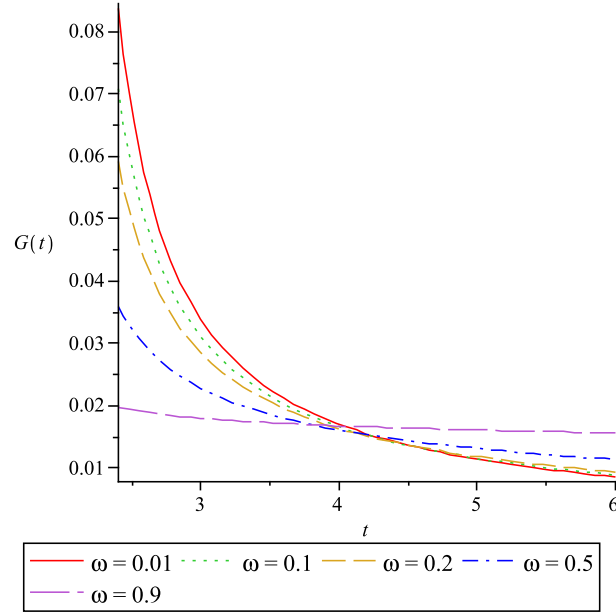Fig. 2. Variation of pressure with time for the given values of ω.

Fig. 3. Variation of time-dependent gravitational constant with time for the given values of ω.
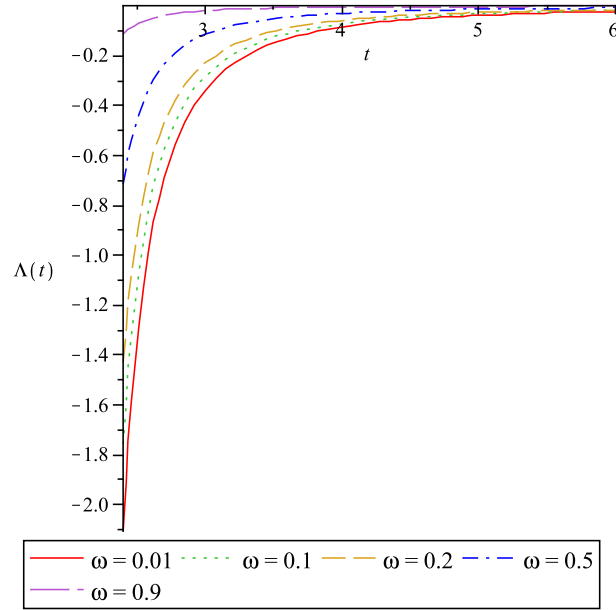

Fig. 4. Variation of time-dependent cosmological constant with time for the given values of ω.